Study of Optimal Guidance Algorithms

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This paper describes and compares a number of indirect, linear, and nonlinear optimal guidance schemes that require information from a precomputed reference trajectory. The methods are based on one step iterative techniques for solution of the nonlinear boundary equations. For each guidance command, the schemes require accurate evaluation of the functions g_i defining the boundary conditions. The derivatives of the functions g_i required in the iterative techniques are obtained cheaply by correcting precomputed derivatives corresponding to the trajectory. The study also includes the Silber-Hunt method, a form of second-variation guidance.

Introduction

The general approach to optimal guidance, considered in this paper, consists of the solution of end constraints $g_i = 0$, where g_i is considered to be a function of t_F and the initial values of the Lagrange multipliers. Values of g_i, corresponding to given values of t_F and the initial multipliers, are obtained by means of numerical integration of the equations of motion and the Euler-Lagrange equations of the calculus of variations. A large class of guidance schemes is embodied in the approach, but in each case g_i is expanded in Taylor series about approximations to t_F and the initial Lagrange multipliers obtained from a reference optimal trajectory. These series are truncated after several terms and set equal to zero. The resulting system of polynomial equations is either inverted to obtain explicit expressions for t_F and the initial multipliers, or solved numerically for these corrections. In all of the schemes, derivatives of g_i with respect to t_F and the initial multipliers are required. These derivatives correspond to the initial state and the reference t_F and the initial multipliers. They may be computed numerically (by integration of differential equations referred to as the equations of variation) or obtained approximately (in ways to be described) from the reference trajectory.

These new nonlinear guidance schemes, unlike linear guidance or the method of Silber and Hunt,[1] are self-correcting, i.e., errors introduced by a drift away from the reference path are removed. It is possible to strengthen the new methods by combining them (in a manner to be discussed) with the method of Silber and Hunt.

The derivation of necessary conditions, by means of the Calculus of Variations (COV), can be found elsewhere[2] and will not be repeated here.

The motion and Euler-Lagrange differential equations describing the optimal paths for minimum fuel consumption are

$$\ddot{x} = \frac{F}{m|\lambda|}\lambda - \frac{\mu}{|x|^3}x$$
$$\ddot{\lambda} = \frac{\mu}{|x|^3}\left(-\lambda + \frac{x\cdot\lambda}{|x|^2}x\right) \tag{1}$$

where x and $\dot{x}$ are the position and velocity vectors with respect to a nonrotating, Earth-centered Cartesian coordinate system, λ and $-\dot{\lambda}$ are the corresponding Lagrange multipliers, μ is the gravitational constant, F is the constant thrust magnitude, $m = m_o - \beta(t - t_o)$, and β is the constant fuel burning-rate magnitude. The subscripts o and F signify initial and final values, respectively. Let ξ represent an N vector of discrete unknown quantities, e.g., missing initial values, final time, and possibly other unspecified quantities, and define y to be an S vector of initial state parameters.

The initial and final end constraints may be represented by the equations

$$f_i(\eta,y,\xi) = 0 \qquad (i = 1, \ldots, N)$$

where $\eta = \eta(y,\xi)$ includes the final states and multipliers. These end constraints are usually geometric end conditions, transversality equations from the COV, and scaling conditions. Let $g_i(y,\xi) = f_i[\eta(y,\xi), y, \xi]$. Then it is desired to solve the equations

$$g_i(y,\xi) = 0 \qquad (i = 1, \ldots, N) \tag{2}$$

for ξ in terms of initial state parameters y. Implicit in Eq. (2) is the solution to the differential Eqs. (1).

To illustrate the notation, consider a minimum fuel, constant burn mission into a prescribed terminal orbit from a specified position and velocity. The initial-state vector has the form $y = (x_o, \dot{x}_o, F/m_o, \beta/m_o)^T$ and t_o is given. Then ξ becomes the 7 vector $(\lambda_o, \dot{\lambda}_o, t_F)^T$. The seven boundary conditions include five geometric terminal conditions, one transversality equation, and a scaling condition.

The method of Silber and Hunt considers Eqs. (2) as identities in y, i.e.,

$$g_i[y, \xi(y)] \equiv 0 \qquad (i = 1, \ldots, N)$$

With the necessary assumptions from the implicit function theorem, the Taylor series expansion of $\xi(y)$ about some nominal $\bar{y}$ is

$$\xi_i(y) = \xi_i(\bar{y}) + \sum_{\alpha=1}^{S}\frac{\partial\xi_i}{\partial y_\alpha}(\bar{y})\Delta y_\alpha + \frac{1}{2}\sum_{\alpha=1}^{S}\sum_{\beta=1}^{S}\frac{\partial^2\xi_i}{\partial y_\alpha\partial y_\beta}(\bar{y})\Delta y_\alpha\Delta y_\beta + \ldots$$

Thus, an explicit formula for ξ in terms of the initial state is immediately obtained. If one proceeds further and determines functions of time for the nominal-state values and derivatives and substitutes these into the aforementioned series, explicit time and state dependent expressions are obtained for ξ.

The Silber-Hunt guidance technique (a form of second-variation guidance) is noniterative in nature and consequently not self-correcting. The method is computationally

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fast but requires a large amount of preflight preparation. The general assumption is that the vehicle will fly in some linear region about the reference trajectory, and hence, a linear series is sufficient or the region is at worst quadratic, hence, a second-order expansion is adequate, and so on.

To visualize the relation between the Silber-Hunt method and iterative techniques, consider a simplified geometric explanation. Let y and ξ be scalar variables. In Fig. 1, a three-dimensional surface $g(y,\xi)$ has been sketched. For simplicity assume that a nominal $\bar{y}$ is zero at some fixed time. Suppose the vehicle is currently at the true state $\bar{y}$. Then it is required to calculate ξ corresponding to point 1. The linear Silber-Hunt method uses the tangent line through $(o,\tilde{\xi})$ in the y, ξ plane to estimate ξ by point 2. Of course, higher-order methods would pass higher order polynomials through $(o,\tilde{\xi})$ in the y, ξ plane. A linear iterative method (e.g., Newton-Raphson) uses the tangent line through the point $[\bar{y},\tilde{\xi}, g(\bar{y},\tilde{\xi})]$ denoted by point 3. The intersection of this line with the y, ξ plane is the estimate ξ^* for ξ. This process can be repeated by using the tangent line through $[\bar{y},\xi^*, g(\bar{y},\xi^*)]$ to obtain a new estimate for ξ.

The computation of the parameters of the approximating curve at point 3 is somewhat time consuming and it must be done at least once for every change in y. However, it seems plausible to approximate them from the corresponding parameters at $(o,\tilde{\xi})$.

Guidance by Solution of Polynomial Equations

In this section the system of nonlinear Eqs. (2) is treated. Here consideration is given to interpolatory iteration functions. Only one iteration (i.e., the solution of one set of polynomials) per guidance command is considered. One may consult Ref. 3 for a discussion of iteration formulas.

Let y be the true state of the space vehicle and let ξ' be an approximation to the solution ξ. Then pass a pth-degree polynomial (i.e., N polynomials in the N variables $\xi_i - \xi_i'$) such that its value at ξ' agrees with g at (y,ξ'). Similarly, constrain the first p-1 derivatives of the polynomials to agree with the first p-1 derivatives of g_i at the point (y,ξ'). This is, of course, equivalent to a truncated Taylor expansion of $g_i(y,\xi)$ about ξ'. Let $\Delta\xi = \xi - \xi'$, $g_i^{(j)} = \partial g_i/\partial\xi_j$, $g_i^{(j,k)} = \partial^2 g_i/\partial\xi_j\partial\xi k$. A Taylor series expansion of $g_i(y,\xi) = 0$ about ξ' yields

$$g_i(y,\xi') + \sum_{j=1}^{N} \Delta\xi_j g_i^{(j)}(y,\xi') + \frac{1}{2!}\sum_{j=1}^{N}\sum_{k=1}^{N} \Delta\xi_j\Delta\xi_k g_i^{(j,k)}(y,\xi') + \ldots = 0 \quad (3)$$

$$(i = 1, 2, \ldots, N)$$

The series must be truncated at some term. The guidance scheme requires solution of the resulting system of polynomial equations for $\Delta\xi$ (e.g., Newton-Raphson). Once the polynomial coefficients have been calculated, the iterative solution of Eq. (3) requires no additional trajectory calculations or numerical integrations.

The derivatives $g_i^{(j)}(y,\xi'), g_i^{(j,k)}(y,\xi'), \ldots$ may be calculated by means of the equations of variation. Although the latter method may be used in numerical studies, it is out of the question (at least in the case of higher derivatives) in onboard implementation of the guidance scheme. Instead, the derivatives may be approximated as follows:

$$g_i^{(j)}(y,\xi') \cong g_i^{j}(\bar{y},\tilde{\xi}) + \sum_{\alpha=1}^{S} \Delta y_\alpha g_{iy_\alpha}{}^{(j)}(\bar{y},\tilde{\xi}) + \sum_{\alpha=1}^{N} (\xi_\alpha' - \tilde{\xi}_\alpha) g_i^{(j,\alpha)}(\bar{y},\tilde{\xi}) \quad (4)$$

$$g_i^{(j,k)}(y,\xi') = g^{(j,k)}(\bar{y},\tilde{\xi})$$

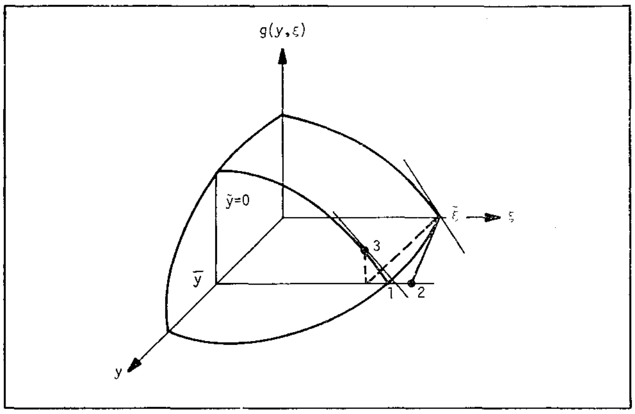

Fig. 1 Geometrical interpretation.

where $\tilde{\xi}$, $\bar{y}$ are the reference values and $\Delta y = y - \bar{y}$. Given y, the reference values $\bar{y}$ and $\tilde{\xi}$ may be determined from the reference trajectory by any of several procedures. Reference 4 discusses one technique, but in implementing the guidance routine a time-to-go criterion is used in this paper.

Numerical calculations of the derivatives, corresponding to an S-IVB injection into circular orbit, indicate that the derivatives do not vary radically as a function of y within a rather large neighborhood of y. On the other hand, the Lagrange multipliers and t_F change appreciably. Furthermore, as long as the functional values, $g_i(y,\xi')$, are computed accurately, it is not necessary to have very accurate higher derivatives in order to compute accurate guidance commands. Therefore, it is considered reasonable to use Eqs. (4) to determine the derivatives.

In the guidance problem the derivatives must be updated from time t_1 to time t_2 as the space flight progresses. This may be done by either of the following two means: 1) numerical integration of the adjoint differential equations[5] forward over the short time interval between t_1 and t_2, and 2) evaluation of polynomials expressing the reference derivatives as functions of time; these polynomials can be determined before flight. The latter method was followed in the preparation of this paper.

A refinement to the guidance schemes previously discussed is the use of the technique of Silber and Hunt[1] to determine a first correction ξ' to ξ for given Δy. Thus,

$$\xi' \cong \tilde{\xi} + \sum_{i=1}^{S} \Delta y_i \tilde{\xi}_{y_i} + \frac{1}{2!}\sum_{i=1}^{S}\sum_{j=1}^{S} \Delta y_i \Delta y_j \tilde{\xi}_{y_i y_j}$$

where $\tilde{\xi}$, $\tilde{\xi}_{y_i}$, etc., are to be evaluated along the reference trajectory. Again the latter derivatives can be expressed as polynomial functions of time for updating or they can be obtained by integrating matrix Riccati equations from the time of one guidance command to the next.[5]

Inversion Formulas for Guidance

The inverse series of g_i can be derived by letting the equations

$$g_i(y,\xi) = w_i \qquad (i = 1, \ldots, N)$$

define ξ implicitly as a function of w_i for fixed y. Then expanding ξ in a Taylor series about $w_i' = g(y,\xi')$ and evaluating the series at $w_i = 0$ gives the resulting inversion. This straight-forward inversion is carried out in Ref. 6 with the

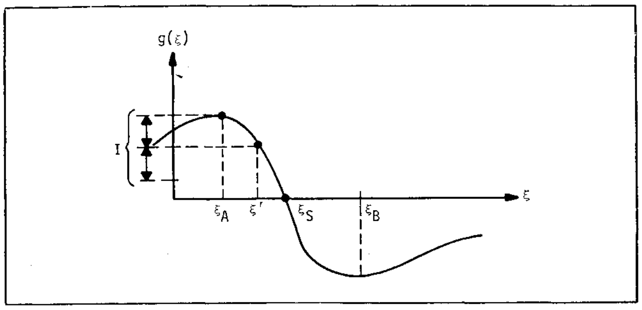

Fig. 2 Regions of convergence.

following results:

$$\Delta\xi = -A^{-1}\left\{g + \frac{1}{2!}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\left(\sum_{\alpha=1}^{N} c_{j\alpha}g_\alpha\right)\times\left(-\sum_{\beta=1}^{N}\sum_{\gamma=1}^{N}\theta_{\beta\gamma}{}^{(k)}g_\beta g_\gamma + \sum_{\beta=1}^{N} c_{k\beta}g_\beta\right)\right]g^{(j,k)} - \frac{1}{3!}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\left[\left(\sum_{\alpha=1}^{N} c_{j\alpha}g_\alpha\right)\left(\sum_{\beta=1}^{N} c_{k\beta}g_\beta\right)\times\left(\sum_{\gamma=1}^{N} c_{l\gamma}g_\gamma\right)\right]g^{(j,j,l)}\right\} \quad (5)$$

where the series has been truncated after third-degree terms in g_α, g_β, and g_γ and where $A = [g_i{}^{(j)}(y,\xi)]$, $C = (c_{ij}) = A^{-1}$,

$$\theta_{\beta\gamma}{}^{(i)} = \frac{\partial^2\Delta\xi_i}{\partial g_\beta \partial g_\gamma} = i\text{th component of}$$

$$\left[-A^{-1}\sum_{j=1}^{N}\sum_{k=1}^{N} c_{j\beta}c_{k\beta}g^{(j,k)}\right]$$

In formula (5), g and its derivatives are to be evaluated at (y,ξ'). The derivatives $g_i{}^{(j)}(y,\xi')$, $g_i{}^{(j,k)}(y,\xi')$, etc., may be determined in the manner discussed earlier.

Comparison of Methods

The discussion of convergence in the large is by necessity intuitive and will be limited to a discussion of the solution of a single equation $g(\xi) = 0$ in one unknown ξ. However, the ideas can be generalized. The desired solution is ξ_s (Fig. 2). A Newton iteration with proper damping (limiting) of the corrections would converge to ξ_s for any initial approximation between ξ_A and ξ_B. The region (not radius) of convergence of an undamped Newton method would be quite a bit smaller (but the rate of convergence usually faster). A second-degree polynomial, passing through the point $[(\bar{\xi},g(\bar{\xi})]$ and having its first and second derivatives equal to those of g at $\xi = \bar{\xi}$, would appear to have a larger region of convergence than Newton's method. Perhaps, in many cases, the region would be nearly as large as that of the damped Newton method. However, inversion about $\bar{\xi}$ would lead to a series which does not converge outside of the interval I indicated in Fig. 2, because the radius of convergence of the series would be less $g(\xi_A) - g(\bar{\xi})$, there being a singular point in the inverse series at $\xi = \xi_A$. Although the inversion formula may give an explicit solution, any advantage this may have is reduced by the limited region of convergence of the inverse series. Some of these intuitive observations are substantiated by numerical results to be given.

We now consider a numerical comparison of the guidance formulas discussed in this paper. The symbol N will represent the degree of the formula used. The problem under numerical study is that of an S-IVB minimum time injection into a 105 naut mile circular orbit from a point 5 miles below the orbit. The initial and final end constraints for the two-dimensional problem are

$$f_1 = x_f\cdot x_f - R_{co}{}^2 = 0 \quad \text{(orbital radius)}$$
$$f_2 = \dot{x}_f\cdot\dot{x}_f - V_{co}{}^2 = 0 \quad \text{(orbital velocity)}$$
$$f_3 = x_f\cdot\dot{x}_f = 0 \quad \text{(orthogonality)}$$
$$f_4 = \lambda_o\cdot\lambda_o - 1 = 0 \quad \text{(scaling)}$$
$$f_5 = \lambda_{10}\dot{x}_{20} - \lambda_{20}\dot{x}_{10} - \dot{\lambda}_{10}x_{20} + \dot{\lambda}_{20}x_{10} = 0 \quad \text{(transversality)}$$

and

$$y = \left(x_{10},\ x_{20},\ \dot{x}_{10},\ \dot{x}_{20},\ \frac{F}{m_o},\ \frac{\beta}{m_o}\right)^T$$

with

$$\xi = (\lambda_{10},\ \lambda_{20},\ \dot{\lambda}_{10},\ \dot{\lambda}_{20},\ t_f)^T$$

Generally speaking, the transversality condition should be imposed at the terminal point, however, in this case the function f_5 is a constant of the motion. Thus, it is required to solve the equations

$$g_i(y,\xi) = 0 \qquad (i = 1,2,\ldots,5)$$

where, for example $g_1(y,\xi) = f_1[\eta(y,\xi),y,\xi] = x_f(y,\xi)\cdot x_f(y,\xi) - R_{co}{}^2$. The reference trajectory (which satisfies, in this case, all boundary conditions) has the following parameters defining it:

x_{10} = 1761674.2 m $\dot{x}_{10}$ = 6546.5205 m/sec
x_{20} = 6314804.0 m $\dot{x}_{20}$ = −1728.0676 m/sec

Final altitude ≐ 105 naut miles

R_{co} = 6565710. m β = 22.0179 mass units/sec (mass flow rate)
$V_{co}{}^2$ = μ/R_{co} c = 4120.193 m/sec (exhaust velocity)
mo = 16645.5 "mass units" (kg-sec²/m) $F = c\beta$

Lagrange multipliers:

λ_{10} = 0.974 $\dot{\lambda}_{10}$ = -0.179×10^{-2}
λ_{20} = 0.228 $\dot{\lambda}_{20}$ = -0.456×10^{-2}
t_F = 170.3 sec

The initial state can be defined by means of an altitude A, a velocity magnitude V, the angle θ between the local horizontal and the velocity vector, and the mass m_o. In order to give the guidance algorithms a severe test, 16 perturbations on the initial reference values of A, V, θ, and m were made.

The perturbations were ±5% in A_o, V_o, and m_o with ±1 degree changes in θ_o. Some of the perturbations are severe enough to throw the initial radius of the trajectory into the 105 naut mile orbit. Table 1 lists the perturbations and also defines the initial state of cases 3, 11, and 14, which are the only three cases to be considered in this paper. These cases were chosen because they typify the results of the study.

The expansion of Silber and Hunt was obtained for the first point of the reference trajectory. The series gives ξ_i explicitly in terms of the perturbations Δy_i. The thrust control angle χ and its time derivative $\dot{\chi}$ are computed directly from the ξ_i. The angle χ and its time derivative $\dot{\chi}$ are measured in degrees with time in seconds. In Table 2, the results of the first ($N = 1$) and second ($N = 2$) order expansions are given. Also the true values are tabulated with the corresponding percent errors.

Table 1 Perturbations of nominal trajectory

Case	A	V	M	θ
3	+	+	+	−
11	−	+	+	−
14	−	−	−	−
Nominal	100 nautical miles	6780.6832 m/sec	16645.597 mass units	0°

Table 2 Silber-Hunt (S-H) expansion

Case		χ	$\dot{\chi}$	t_F	% Error χ	% Error $\dot{\chi}$	% Error t_F
3	True values	65.7	0.30	127.1			
	Nominal	76.8	0.23	170.3	−17.0	23.5	−34.0
	S-H N = 1	66.8	0.28	130.0	−1.7	6.3	−3.1
	S-H N = 2	65.1	0.30	128.9	0.9	1.1	−1.4
11	True values	39.9	0.16	167.9			
	Nominal	76.8	0.23	170.3	−92.5	−48.9	−1.5
	S-H N = 1	40.7	0.18	146.2	−1.9	−16.6	12.9
	S-H N = 2	28.5	0.07	161.9	28.5	52.3	3.6
14	True values	54.2	0.15	229.7			
	Nominal	76.8	0.23	170.3	−41.7	−52.2	25.8
	S-H N = 1	43.8	0.13	227.9	19.3	12.6	0.8
	S-H N = 2	58.2	0.09	235.3	−7.3	43.1	−2.5

Table 4 Guidance formulas with derivatives obtained by correcting reference derivatives and using nominal starting values

Case		χ	$\dot{\chi}$	t_F	% Error χ	% Error $\dot{\chi}$	% Error t_F
3	Polynomial solution						
	N = 1	69.9	0.27	128.1	−6.4	12.0	−0.8
	N = 2	68.9	0.31	125.5	−5.0	−3.4	1.2
11	Polynomial solution						
	N = 1	40.8	0.17	144.8	−2.1	−14.1	13.7
	N = 2	39.2	0.16	156.3	1.7	−7.3	6.9
14	Polynomial solution						
	N = 1	33.7	−0.08	232.1	37.9	46.9	−1.1
	N = 2	50.1	0.15	227.8	7.7	3.6	0.8

The guidance formulas described require the computation of various derivatives. In Table 3 the derivatives in the guidance formulas are calculated by integrating the equations of variation. The resulting χ and $\dot{\chi}$ for the perturbations are tabulated along with the percentage errors. The inversion formulas of orders one, two, and three, along with the solutions of the first-, second-, and third-degree polynomials, are given. In addition the results of a Newton-Raphson procedure at the end of two iterations are listed.

In Table 4, the derivatives of the guidance formulas were computed by correcting corresponding derivatives from the reference trajectory. The resulting χ and $\dot{\chi}$ are listed for each of the cases. Here only the $N = 1$ and $N = 2$ orders of the polynomial formulas are considered. The data in Tables 3 and 4 were obtained by using the initial point of the nominal trajectory as starting values. Comparing the values in Table 4 to the corresponding ones in Table 3, one observes that the results differ very little.

Study of tabulated errors for the 16 cases originally considered clearly indicates the superiority of the polynomial solutions over the inversion formulas. Quantitatively, for example, the percentage errors in the second-degree polynomial solutions for χ exceeded 15% in only 2 cases out of 16 while the second-order inversion formulas exceeded 15% in 7 cases.

Based on these results a guidance algorithm employing second-degree interpolatory iteration was chosen for more detailed study.

Table 3 Guidance formulas with derivatives numerically integrated and using nominal starting values

Case		χ	$\dot{\chi}$	t_F	% Error χ	% Error $\dot{\chi}$	% Error t_F
3	Inversion						
	N = 1	69.9	0.252	128.0	−6.3	16.5	−0.7
	N = 2	67.0	0.286	126.7	−1.9	5.2	0.3
	N = 3	66.0	0.294	126.8	−0.5	2.8	0.2
	Polynomial						
	N = 1	69.9	0.252	128.0	−6.3	16.5	−0.7
	N = 2	68.4	0.308	125.9	−4.1	−1.7	0.9
	N = 3	65.0	0.304	126.8	1.0	−0.7	0.2
	Newton-Raphson	65.6	0.303	126.7	0.1	−0.3	0.3
11	Inversion						
	N = 1	40.8	0.178	144.6	−2.1	−14.6	13.8
	N = 2	27.4	0.064	161.5	31.4	58.5	3.8
	N = 3	32.3	0.097	171.8	19.1	37.3	−2.3
	Polynomial						
	N = 1	40.8	0.178	144.6	−2.1	−14.6	13.8
	N = 2	38.4	0.159	157.2	3.8	−2.4	6.3
	N = 3	39.8	0.160	163.7	0.4	−3.0	2.5
	Newton-Raphson	39.1	0.146	168.1	2.1	6.0	−0.1
14	Inversion						
	N = 1	33.6	0.081	232.0	37.9	46.8	−1.0
	N = 2	77.8	−0.247	243.0	−43.6	262.9	−5.8
	N = 3	66.3	0.289	216.5	−22.2	−90.6	5.7
	Polynomial						
	N = 1	33.6	0.081	232.0	37.9	46.8	−1.0
	N = 2	50.6	0.150	227.9	6.6	1.7	0.7
	N = 3	54.2	0.154	228.2	0.0	−1.7	0.6
	Newton-Raphson	53.6	0.193	198.0	1.0	−27.5	13.7

Recall that the starting values used in Table 4 were the nominal values, and were generally very poor, which can be seen by examining the errors in Table 2. Instead, the second-order expansion of Silber and Hunt was used to generate starting values. Then the second-degree polynomials were obtained using corrected nominal derivatives. Table 5 contains the results of this procedure under the heading "First Guidance Command." In comparing the percentage errors in χ and $\dot{\chi}$ of Table 5 to those of the second-order expansion of Silber and Hunt in Table 2 one sees that the error is reduced in every case. Concerning the errors in t_F, it was observed that in most cases the error of the guidance formulas and the Silber-Hunt expansion is acceptable. However, in two of the 16 cases considered (including case 11) the error in t_F of the guidance formula was much greater than the corresponding Silber-Hunt error. For this reason, it was decided to return the Silber-Hunt estimate of time-to-go on the first guidance cycle. In order to investigate the initial behavior of the guidance package, a second guidance cycle was computed with no change in the initial state y. The results are tabulated in Table 5 under the heading second guidance command. All errors were more than acceptable.

The results of the numerical study are compared qualitatively on the basis of accuracy and convergence in Table 6. The ratings are determined by computing a weighted percent error by weighting χ and $\dot{\chi}$ two and t_F one. The reasoning here being that χ and $\dot{\chi}$ are of direct initial importance, whereas the beginning values of t_F are merely indicators of future state and do not affect current action. It is clear from

Table 5 Guidance formulas with derivatives obtained by correcting reference derivatives and using Silber-Hunt starting values

Case		χ	$\dot{\chi}$	t_F	% Error χ	% Error $\dot{\chi}$	% Error t_F
3	Polynomial solution						
	First guidance command						
	N = 1	65.6	0.30	127.2	0.2	0.0	0.1
	N = 2	65.6	0.30	127.2	0.1	0.0	0.1
	Second guidance command						
	N = 1	65.6	0.30	127.1	0.0	0.0	0.0
	N = 2	65.6	0.30	127.1	0.0	0.0	0.0
11	Polynomial solution						
	First guidance command						
	N = 1	47.3	0.13	214.9	−18.6	14.8	−28.0
	N = 3	14.2	−0.07	322.2	64.5	146.1	−92.0
	Second guidance command						
	N = 1	34.7	0.13	157.8	13.0	16.3	6.0
	N = 2	37.2	0.15	164.9	6.7	7.0	1.7
14	First guidance command						
	N = 1	51.6	1.4	225.7	4.8	8.7	1.7
	N = 2	52.9	0.14	234.1	2.5	5.5	−1.9
	Second guidance command						
	N = 1	53.7	0.15	231.1	1.0	1.0	−0.6
	N = 2	53.8	0.15	231.1	0.8	0.8	−0.6

Table 6 **Qualitative comparison of the guidance formulas**

Method	Ratings		
	A	B	C
Nominal	0	0	16
Silber-Hunt expansion			
$N = 1$	3	8	5
$N = 2$	4	8	4
Inversion (integrated derivatives, nominal starting values)			
$N = 1$	0	9	7
$N = 2$	4	6	6
$N = 3$	5	3	7
Polynomial (integrated derivatives, nominal starting values)			
$N = 1$	0	9	7
$N = 2$	10	4	2
$N = 3$	13	0	3
Damped Newton-Raphson (2 iterations)	11	4	1
Polynomial (corrected nominal derivatives, nominal starting values)			
$N = 1$	0	9	7
$N = 2$	10	4	2
Polynomial (corrected nominal derivatives, S-H starting values)			
$N = 1$	10	3	3
$N = 2$	12	2	2
Polynomial (second guidance command)			
$N = 1$	14	1	1
$N = 2$	14	2	0
E = weighted error magnitude $E \leq 5\%$ A; $5\% < E \leq 15\%$ B; $E > 15\%$ C			

Table 6 that the use of the Silber-Hunt expansion greatly improves the convergence of the guidance formulas.

In summary the main conclusions of this numerical study are listed:

A) The $N = 1$ polynomial solutions improved the Silber-Hunt approximations of χ and $\dot{\chi}$ in every case. The advantage of this combination of methods is obvious. Uncorrected Silber-Hunt (second variation) solutions would probably not be of sufficient accuracy for guidance.

B) The polynomial solutions were obviously more effective than the inversion formulas. Thus, the apparent advantage of closed form solutions is misleading.

C) The $N = 2$ polynomial solution on the first guidance command gave little improvement over the $N = 1$ polynomials. (In cases 11 and 12 the iteration on the $N = 2$ polynomials did not converge.)

D) The $N = 2$ polynomial solution gave significant improvement over the $N = 1$ polynomial solution on the second guidance command in cases 11 and 12 where it was most needed. However, in view of conclusion C, the use of $N = 2$ is held open (see below).

E) The Silber-Hunt second-order terms improved the first-order terms, especially t_F. However, the $N = 2$ terms failed to improve the $N = 1$ terms in a few cases (notably cases 11 and 12) and even deteriorated the $N = 1$ estimates. Considering the additional computing time and storage requirements for $N = 2$, the advisability of using it for calculating approximate multipliers is open to question.

Guidance Algorithm under Simulated Flight Conditions

A complete guidance routine based on the methods discussed previously was programed and employed in a space flight simulation computer program which simulated the S-IVB flight in two dimensions. The numerical results of these studies are given in Tables 7–9. The $N = 1$ and $N = 2$ methods were applied to each of three missions, corresponding to cases 3, 11, and 14 previously discussed. Between guidance commands (given at 8 sec intervals) χ was assumed to be linear. The payload delivered was compared to the greatest possible payload corresponding to an exact COV solution. It may be seen from the tables that the order of magnitude of the payload differences is 100 lb. Studies have indicated that this magnitude would probably be reduced if smaller guidance cycle times were employed.

The tables also show the errors in altitude and velocity magnitude at cutoff time $t_{co} = t_F$. The flight path angle θ at time t_F was always within $\frac{1}{10}$ of a degree of the correct angle, 90°.

The tables indicate that there is no advantage to the use of the $N = 2$ polynomial solution over the $N = 1$ solution. The reason for this may be that the values of the second derivatives employed are merely those corresponding to the reference flight path. (The first derivatives, however, consist of corrected reference derivatives.) Therefore, the second derivatives may be too inaccurate to improve the $N = 1$ polynomial solution. The tables also indicate that the second-order Silber-Hunt method gives little improvement over the first-order method. As a result of these studies, the second-order terms have been removed from the guidance routine except in the calculation of final time t_F on the first guidance command. Second derivatives are still used to correct reference first derivatives to account for deviations from the reference state and Lagrange multiplier variables. It is now this technique for obtaining first derivatives that largely distinguishes the guidance method from IBM's OPT guide.[7]

Conclusions

The numerical algorithms discussed in this paper add self-correcting features to Silber-Hunt guidance. However, the techniques are still dependent on proximity to a reference path and require storage of coefficients of polynomial functions of time. The comparison of the polynomial solutions and the inversion formulas on an S-IVB type trajectory clearly indicates the superiority of the polynomial solutions. One of the more important and unique features of the guidance algorithm was the rapid technique used for computing approximate first and second derivatives of functions of the final state with respect to current Lagrange multipliers; the first derivatives were approximated by correcting reference derivatives to account for deviations from the current nominal state and nominal Lagrange multipliers. This technique for

Table 7 **Application of guidance scheme to case 3**

	Mass (IB) at t_F	COV mass (IBS) at t_F	Differential Mass (IBS)	$R - R_{co}$ (meters)	$V - V_{co}$ (m/sec)
$N = 2$ Polynomial, $N = 2$ Silber-Hunt	317405.1	317487.6	82.5	79.0	−0.6
$N = 1$ Polynomial, $N = 2$ Silber-Hunt	317408.2	317487.6	79.4	80.0	−0.7
$N = 1$ Polynomial, $N = 1$ Silber-Hunt	317407.6	317487.6	80.0	80.0	−0.7

Table 8 Application of guidance scheme to case 11

	Mass (IBS) at t_F	COV mass (IBS) at t_F	Differential mass (IBS)	$R - R_{co}$ (meters)	$V - V_{co}$ (m/sec)
$N = 2$ Polynomial, $N = 2$ Silber-Hunt	297692.9	298091.1	398.2	279	−10.8
$N = 1$ Polynomial, $N = 2$ Silber-Hunt	297690.5	298091.1	400.6	255	−10.4
$N = 1$ Polynomial, $N = 1$ Silber-Hunt	297755.8	298091.1	235.3	250	−10.1

Table 9 Application of guidance scheme to case 14

	Mass (IBS) at t_F	COV Mass (IBS) at t_F	Differential mass (IBS)	$R - R_{co}$ (meters)	$V - V_{co}$ (m/sec)
$N = 2$ Polynomial, $N = 2$ Silber-Hunt	232565.5	232651.4	85.9	92.0	−1.2
$N = 1$ Polynomial, $N = 2$ Silber-Hunt	232565.5	232651.4	85.9	87.0	−1.3
$N = 1$ Polynomial, $N = 1$ Silber-Hunt	232568.8	232651.4	82.6	87.0	−1.3

finding approximate derivatives led to guidance commands which did not vary much from the guidance algorithm utilizing true derivatives. Included in the numerical study was an independent use of the expansion of Silber and Hunt. The results showed clearly that a combination of the Silber-Hunt expansion with the polynomial solutions proved much more useful than either taken individually. The guidance formulas described were of arbitrary order. However, the numerical results indicate that the second or higher-order formulas are of questionable utility.

A more detailed description of the studies discussed in this paper may be found in Ref. 8.

References

[1] Silber, R. and Hunt, R. W., "Space Vehicle Guidance—A Boundary Value Formulation," TM X-53059, June 1966, NASA.

[2] Leitman, G., ed., *Optimization Techniques*, Academic Press, New York, 1962.

[3] Traub, J. F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[4] Powers, W. F., "A Method For Comparing Trajectories in Optimum Linear Perturbation Guidance Schemes," *AIAA Journal*, Vol. 6, No. 12, Dec. 1968, pp. 2451–2452.

[5] Athans, M. and Falb, P. L., *Optimal Control*, McGraw-Hill, New York, 1966.

[6] Andrus, J. F., "Explicit Solutions to Problems of Optimal Guidance," Rept. TR-792-8-303, Summary Rept for NASA Contract NAS8-20082, Jan. 1968, Northrop Corporation, Huntsville, Ala.

[7] Brown, K. R., and Johnson, G. W., "Real-Time Optimal Guidance," *IEEE Transactions on Automatic Control*, Vol. AC–12, No. 5, Oct. 1967.

[8] Andrus, J. F., Burns, I. F., and Woo, J. Z., "Nonlinear Optimal Guidance Algorithms," Rept. TR-792-9-547, Interim Rept. for NASA Contract NAS12-500, May 1969, Northrop Corporation, Huntsville, Ala.